Chapter 3 Notes

Sections 3.3/3.4

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Reference: A First Course in Modular Forms (Diamond and Shurman)

3 Dimension Formulas

3.2 Motivation

Before we look into differentials and Riemann-Roch to see their applications to Modular forms, let us take a look into why we are so motivated to use them. Consider a function f which is an automorphic of weight k. Then we know for $\tau \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$f(\gamma(\tau)) = j(\gamma,\tau)^k f(\tau) = (c\tau + d)^k f(\tau). \tag{1}$$

Where Γ is any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Although there is no discussion in the text, some equations throughout this chapter can be relaxed to the assumption that Γ is only a finite index subgroup. For our purposes, we will stick with the stricter condition. Previously mentioned in Section 1.1 of the text, $d(\gamma(\tau)) = (c\tau + d)^{-2} d\tau$. A natural result arises for f being any automorphic form of weight $k/2$ where k is a positive, even integer

$$\begin{aligned} f(\gamma(\tau))(d\gamma(\tau))^{k/2} &= j(\gamma,\tau)^k f(\tau)(j(\gamma,\tau)^{-2}(d\tau)^{k/2} \\ &= j(\gamma,\tau)^k f(\tau) j(\gamma,\tau)^{-k}(d\tau)^{k/2} \\ &= f(\tau)(d\tau)^{k/2}. \end{aligned}$$

Demonstrating $f(\tau)(d\tau)^{k/2}$ is in fact $\Gamma-$invariant. It is the goal of this section to show that the automorphic forms of even weight have a natural $\mathbb{C}-$vector space isomorphism with meromorphic differentials of half the weight. Thus, the need to understand these differentials and the use of theorems in geometry such as Riemann-Roch is, in essence, to help understand the subspaces of automorphic forms (which would include the space of modular forms).

3.3 Meromorphic Differentials

We begin with a definition of the main topic of this section

Definition 1. Let $V \subset \mathbb{C}$ be open and $n \in \mathrm{N}$. The *meromorphic differentials* on V of degree n are

$$\Omega^{\otimes n}(V) = \{f(q)(dq)^n : f \text{ meromorphic on } V, q \text{ is the variable on } V\}.$$

A seemingly obvious yet important remark is that one must keep track of which coordinate system they work in while considering meromorphic differentials. We will be looking into Riemann Surfaces which have local charts. We immediately notice that $\Omega^{\otimes n}(V)$ forms a $\mathbb{C}-$vector space under the usual operations

$$f(q)(dq)^n + g(q)(dq)^n = (f+g)(q)(dq)^n \text{ and } cf(q)(dq)^n = (cf)(q)(dq)^n \text{ for } c \in \mathbb{C}.$$

If we consider the space of meromorphic differentials on V denoted $\Omega(V) = \bigoplus_{n \in \mathrm{N}} \Omega^{\otimes n}(V)$ and add the operation $(dq)^n (dq)^m = (dq)^{n+m}$ we get a ring structure.

Notice the initial definition is only defined locally (for open sets). We would like to figure out a way to construct a globally defined meromorphic differential for a Riemann Surface X. to do this, we need to patch together local meromorphic differentials, meaning we need to see what happens with transition maps.

Remark 1. Every holomorphic function $\varphi : V_1 \to V_2$ between two open subsets of $\mathbb{C}$ induces a *pullback* map $\varphi^* : \Omega^{\otimes n}(V_2) \to \Omega^{\otimes n}(V_1)$ defined by

$$\varphi^*(f(q_2)(dq_2)^n) = f(\varphi(q_1))\varphi'(q_1)(dq_1)^n. \tag{2}$$

Some important properties of the pullback that can be verified straight from the definition are the following:

Proposition 1. Let $\varphi_1 : V_1 \to V_2$ and $\varphi_2 : V_2 \to V_3$ be two holomorphic maps between open sets V_1, V_2, V_3. Then $(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$.

Proof. Notice that $\varphi_2 \circ \varphi_1$ is a holomorphic map from V_1 to V_3. by the definition of the pullback, for $f(q_3)(dq_3)^n \in \Omega^{\otimes n}(V)$ we have

$$\begin{aligned}
(\varphi_2 \circ \varphi_1)^*(f(q_3)(dq_3)^n) &= f(\varphi_2(\varphi_1(q_1)))(\varphi_2 \circ \varphi_1)'(q_1)^n (dq_1)^n \\
&= f(\varphi_2(\varphi_1(q_1)))(\varphi_2'(\varphi_1(q_1))\varphi_1'(q_1))^n (dq_1)^n \\
&= \varphi_1^*(f\varphi_2(q_2)(dq_2)^n) \\
&= \varphi_1^*(\varphi_2^*(f(q_3)(dq_3)^n)).
\end{aligned}$$

□

Two ideas used in the proof: a meromorphic function composed with a holomorphic function is meromorphic and chain rule for complex holomorphic functions. There are a few other properties that come straight from the definition.

Proposition 2. If $V_1 \subset V_2$ and $i : V_1 \to V_2$ is the inclusion map then $i^*(\omega) = \omega|_{V_1}$ for all $\omega \in \Omega^{\otimes n}(V_2)$.

Proposition 3. If φ is a holomorphic bijection then $(\varphi^{-1})^* = (\varphi^*)^{-1}$

Proposition 4. If π is a holomorphic surjection, then π^* is injective.

Recall from complex analysis that any holomorphic bijection has a holomorphic inverse function. Equipped with these tools, we can define a global meromorphic differential for a general Riemann Surface.

Definition 2. Let X be a Riemann Surface with charts $\varphi_j : U_j \to V_j$ with $U_j \subset X, V_j \subset \mathbb{C}, j \in J$ for some indexing set J. A *meromorphic differential* on X of degree n is a collection of local meromorphic differentials of degree n

$$(\omega_j)_{j\in J} = \prod_{j\in J} \Omega^{\otimes n}(V_j)$$

that is *compatible*. That is, let $V_{j,k} = \varphi_j(U_j \cap U_k)$ and $\varphi_k(U_j \cap U_k)$ then the transition map $\varphi_{k,j} : V_{j,k} \to V_{k,j}$ induces the relationship

$$\varphi^*_{k,j}(\omega_k|_{V_{k,j}}) = \omega_j|_{V_{j,k}}.$$

At first glance, this definition can seem quite daunting. Let's work through an example to see how to apply such a definition.

Example 1. Take a complex tori (which can be identified with a complex elliptic curve) $\mathbb{C}/\Lambda$ where Λ is a $\mathbb{Z}-$lattice in $\mathbb{C}$. We have the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ and we define the charts on the torus $\{U_j, \varphi_j\}_{j\in J}$ to be the local homeomorphic inverse to the natural projection. One can reason through the fact that any transition map must be of the form $z \mapsto z + \lambda$ where $\lambda \in \Lambda$. Geometrically, we have the following diagram

Using the definition above, we notice $V_{j,k}$ and $V_{k,j}$ are identified as the same set mod Λ. Thus, for a collection of local meromorphic differentials $\omega = (dz_j)_{j\in J}$ we have

$$\varphi^*_{k,j}((dz_k)^n) = \varphi'_{k,j}(z_j)^n(dz_j)^n = (dz_j)^n$$

since the derivative of any transition map will be 1. Thus, we have a globally defined meromorphic differential dz on the torus.

Much like local meromorphic differentials, the set $\Omega^{\otimes n}(X)$ of meromorphic differentials of degree n on X has a $\mathbb{C}-$vector space structure and

$$\Omega(X) = \bigoplus_{n\in \mathrm{N}} \Omega^{\otimes n}(X)$$

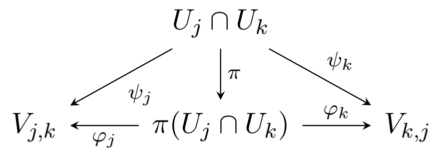
forms a ring.

Now consider the modular space $X(\Gamma) = \Gamma\backslash\mathcal{H}^*$. We want to make sense of meromorphic differentials which are globally defined in this space. Once again consider the natural projection map $\pi : X \to X(\Gamma)$. A question one could consider is what is the pullback map? Is this map a natural pullback? It is discussed in Chapter 2 the construction of charts for the space $X(\Gamma)$ which are $\{\pi(U_j), \varphi_j\}_{j\in J}$ where U_j is a neighborhood of either $\tau \in \mathcal{H}$ or a cusp $s_j \in \mathbb{Q} \cup \{\infty\}$ and the local coordinate map $\varphi_j : \pi(U_j) \to V_j \subset \mathbb{C}$ has the property that the function $\psi : U_j \to V_j$ is the composite $\varphi_j \circ \pi$. So ψ is defined as the identifying action of π but then maps out to $\mathbb{C}$.

Let $(\omega_j)_{j\in J} \in \Omega^{\otimes n}(X(\Gamma)), U'_j = U_j \cap \mathcal{H}$ and $V'_j = \psi(U_j)$ and $\omega'_j = w_j|_{V'_j}$. Then the pullback map π^* is locally defined as

$$\pi^*(\omega)|_{U'_j} = \psi(\omega'_j) \text{for all} j \in J.$$

Note that this definition only gives us what the pullback map is locally defined to be. However, with a little more work, we can show this construction can give a globally defined meromorphic differential on $X(\Gamma)$. Consider the commutative diagram



The transition map $\varphi_{k,j} : V_{j,k} \to V_{k,j}$ is defined by

$$\varphi_{k,j} = (\varphi_k \circ \varphi_j^{-1})|_{V_{j,k}}$$

and

$$\varphi_{k,j} \circ \psi_j = \psi_k \text{ on } U_j \cap U_k.$$

If we let $V'_{j,k} = \psi_j(U'_j \cap U'_k)$ and $V_{k,j} = \psi_k(U'_j \cap U'_k)$ then

$$\pi^*(\omega)|_{U_j \cap U_k} = \psi_k^*(\omega_k|_{V_{j,k}}) = \psi_j^*(\pi^*(\omega_k|_{V'_{k,j}})) = \psi_j^*(\omega_j|_{V_{j,k}})$$

by compatibility since $(\omega_j)_{j\in J} \in \Omega(X(\Gamma))$. The pullbacks have a common $f(\tau)(d\tau)^n|_{U'_j \cap U'_k}$ factor in them so the pullback

$$\pi^*(\omega) = f(\tau)(d\tau)^n \text{ is well defined.}$$

Moreover, since this $f(\tau)(d\tau)^n$ comes from a $\Gamma-$invariant space, then f itself must be $\Gamma-$invariant. That is,

$$\begin{aligned} f(\tau)(d\tau)^n &= f(\gamma(\tau))(d\gamma(\tau))^n \\ &= f(\gamma(\tau))(\gamma'(\tau)^n)(d\tau)^n \\ &= f(\gamma(\tau))(j(\gamma,\tau)^{-2n}(d\tau)^n \\ &= (f[\gamma]_{2n})(\tau)(d\tau)^n \end{aligned}$$

demonstrating that f is weakly modular of weight $2n$. We want to see if f is an automorphic form of weight $2n$. For this we need to check that f is meromorphic at ∞. Let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and $s = \infty$). The local map (as seen in Chapter 2) can be expressed as $\psi = \rho \circ \delta$ for $\rho = e^{\frac{2\pi i \delta(\tau)}{h}}$ where h is the width of s and δ is the transformation which sends s to ∞. Clearly, $\delta = \alpha^{-1}$ by how we defined s and using the additional fact that ω is meromorphic at ∞ we see $\omega|_V$ takes the form $g(q)(dq)^n$ for some g which is meromorphic at 0. We will construct f on the set $U - \{s\}$ by

$$\begin{aligned} \psi(\omega|_{V-\{0\}}) &= \delta^*\rho^*(g(q)(dq)^n) \\ &= \delta^*(g(\rho(z))\rho'(z)^n(dz)^n) \\ &= g(\rho(\delta(\tau)))\rho'(\delta(\tau))^n\delta'(\tau)^n(d\tau)^n \\ &= g\left(e^{\frac{2\pi i\delta(\tau)}{h}}\right)\left(\frac{2\pi i}{h}e^{\frac{2\pi i\delta(\tau)}{h}}\right)^n j(\delta,\tau)^{-2n}(d\tau)^n. \end{aligned}$$

Letting $q = e^{\frac{2\pi i\delta(\tau)}{h}}$ then $\psi(\omega|_{V-\{0\}}) = g(q)q^n\left(\frac{2\pi i}{h}\right)^n[\delta]_{2n}$. Thus, we verify f is meromorphic at ∞ since

$$f = \tilde{f}[\delta]_{2n} \text{ where } \tilde{f} = g(q)q^n\left(\frac{2\pi i}{h}\right)^n.$$

A natural question to ask after this calculation is how can we reverse it. That is, can we create an invertible mapping between the space of Automorphic forms of even weight and the meromorphic differentials on $X(\Gamma)$ of half the degree. To do this, we need to use the following remark and proposition

Remark 2. Every meromorphic differential ω of degree n on $X(\Gamma)$ pulls back to $\pi^*(\omega) = f(\tau)(d\tau)^n$ which is a meromorphic differential on the upper half plane and f is an automorphic form of weight $2n$ with respect to Γ.

Proposition 5. A collection $(\omega_j)_{j\in J}$ of local meromorphic differentials is compatible if and only if its local elements pull back to a restriction of some $f(\tau)(d\tau)^n \in \Omega^{\otimes n}(\mathcal{H})$ with $f \in \mathcal{A}_{2n}(\Gamma)$.

Thus, in order to reverse the calculation and create a desired isomorphism between automorphic forms and meromorphic differentials the goal is to take $f \in \mathcal{A}_{2n}(\Gamma)$ and construct a meromorphic differential $\omega(f) \in \Omega^{\otimes n}(X(\Gamma))$ such that $\pi^*(\omega) = f(\tau)(d\tau)^n$. The previous proposition allows us to simply construct local differentials that will pull back to restrictions on $f(\tau)(d\tau)^n$. Earlier in the section (and in chapter 2) we looked at the charts as the composite map $\psi_j = \rho_j \circ \delta_j$ which moves from τ space to z space and finally to q space. We defined δ_j to be in $\mathrm{GL}_2(\mathbb{C})$ so we move from τ space to z space with ease knowing δ_j is an invertible linear transformation. Define

$$(f[\gamma]_k)(\tau) = (\det(\gamma))^{k/2} j(\gamma,\tau)^{-k} f(\gamma(\tau)) \text{ where } j(\gamma,\tau) = c\tau + d$$

as before but now extending the definition for all matrices in $\mathrm{GL}_2(\mathbb{C})$. Equipped with these new definitions we look at the set $U_j' = U_j \cap \mathcal{H}$ and notice that $f(\tau)(d\tau)^n$ is the pullback of λ_j where λ_j is the pull forward of $f(\tau)(d\tau)^n$ using the inverse matrix δ_j^{-1}. A remark about this calculation is that λ_j is $\delta_j \Gamma \delta_j^{-1}$ invariant coming form $f(\tau)(d\tau)^n$. Pushing into q-space needs to be treated with care since ρ is not necessarily always inveritble. Fulton and Harris split into two cases: dealing with only sets in the upper half plane and when you have a cusp. The difference comes when looking at the local forms of each ψ_j in each case. Everything follows quite naturally from the definitions which we established in chapter 2. This calculation on page 81 in the text gives us a ω_j which locally pulls back to $f(\tau)(d\tau)^n$. We invoke the previous proposition to get the main theorem for this section

Theorem 1. Let $k \in \mathrm{N}$ be even and let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The map

$$\omega : \mathcal{A}_{2k}(\Gamma) \to \Omega^{\otimes k/2}(X(\Gamma))$$

sending f to $(\omega_j)_{j\in J}$ is an isomorphism of complex vector spaces.

Of course this (ω_j) collection is the one constructed before that pulls back to $f(\tau)(d\tau)^n \in \Omega^{\otimes k/2}(\mathcal{H})$. An immediate corollary of this (which will be used in the next section for a calculation) is the following

Corollary 1. Let $k \in \mathrm{N}$ be positive and even, then $\mathcal{A}_k(\Gamma)$ takes the form $\mathbb{C}(X(\Gamma))f$ where $\mathbb{C}(X(\Gamma)$ denotes the field of meromorphic functions on $X(\Gamma)$ and f is any nonzero element of $\mathcal{A}_k(\Gamma)$. Moreover, by the isomorphism above, $\Omega^{\otimes k/2}(X(\Gamma)) = \mathbb{C}(X(\Gamma))\omega(f)$ for such k.

To end the section, there are remarks on the vanishing order for ω and an explicit formula is given. For the purposes of what we expect to use meromorphic differentials for (and in the context of the next section), it is of less importance than the theorem above. Thus, we conclude the initial study of these meromorphic differentials.

3.4 Divisors and Riemann-Roch

The previous section seems of little use to studying modular forms. However, one must realize that the isomorphism create between spaces of automorphic forms and meromorphic differentials now leads to analyzing the subspaces $\mathcal{M}_k(\Gamma)$ and even $\mathcal{S}_k(\Gamma)$ as subspaces of meromorphic differentials. Now we use theorems of complex geometry in order to conclude some characteristics of these spaces. To do this we introduce the following definitions:

Definition 3. A *divisor* on X is a formal sum on the points of X,

$$D = \sum_{x \in X} n_x \cdot x, n_x \in \mathbb{Z}$$

where all but finitely many $n_x = 0$.

In this definition and for the rest of this section, we assume that X is a compact Riemann surface. Although in the context of looking into modular forms, we might imagine dealing with $X(\Gamma)$ for some congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Some trivial examples of divisors are $D = p, D = p+q, D = 2p - q$ for $p, q \in X$ where $p \neq q$.

One can notice immediately that the set $\mathrm{Div}(X)$ which are the divisors of X form the free abelian group on the points of X. That is, for divisors $D = \sum_{x \in X} n_x \cdot x$ and $D' = \sum_{x \in X} n'_x \cdot x$ we have

$$D + D' = \sum_{x \in X} (n_x + n'_x) \cdot x.$$

For convention purposes, we say $D \geq D'$ if $n_x \geq n'_x$ for all $x \in X$.

Definition 4. The degree of a divisor is defined as $\deg(D) = \sum n_x$

Using one of the examples above, for $p, q \in X$ and $p \neq q$ we have $D = p + q$ being a divisor of degree 2. The divisor $D = p$ has degree 1. It can be shown that the map $D \mapsto \deg(D)$ is a group homomorphism from $\mathrm{Div}(X)$ to $\mathbb{Z}$.

We associate each meromorphic function on X a divisor defined by the order of vanishing at each point. Let $f \in \mathbb{C}(X)^*$, we call

$$\mathrm{div}(f) = \sum_{x \in X} \nu_x(f) \cdot f$$

the *principal divisor* of f. This creates yet another morphism from the field of meromorphic functions on X to $\mathrm{Div}(X)$. We have the following proposition that comes from complex anlaysis to aid us in the study of divisors.

Proposition 6. For every $f \in \mathbb{C}(X)^*, \mathrm{div}(f) = 0$.

Proof. □